Contextuality, “All-vs-Nothing” Argument, and Cohomology

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Based on joint works with
Samson Abramsky, Rui Soares Barbosa, Giovanni Carù,
Nadish de Silva, Ray Lal, and Shane Mansfield

Algebraic Structures in Quantum Computation
University of British Columbia
May 25, 2017

Outline

I "Sheaf approach" to contextuality (Abramsky-Brandenburger 2011)

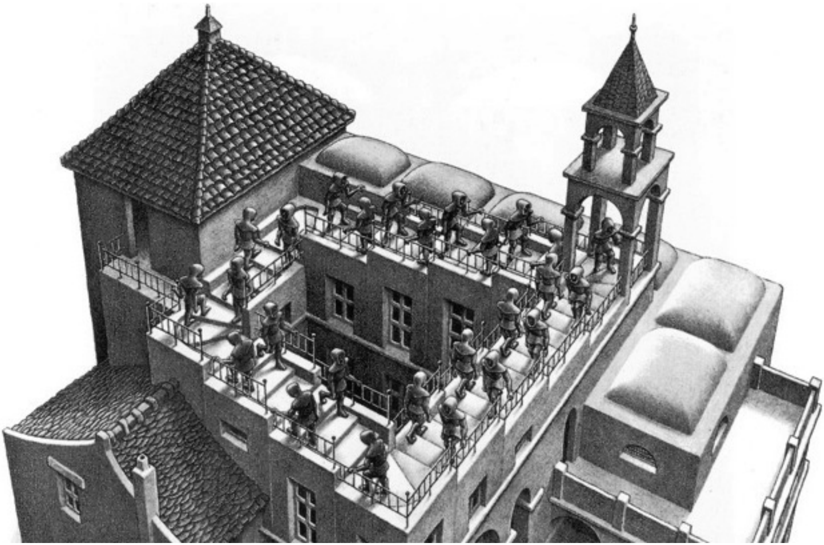
1. Review (of a simplicial-complex formulation)
   (ABKLM 2015, Kishida 2016)
2. Relation to other approaches, e.g. Spekkens
   (ABKLM 2016, Wester 2017, Mansfield n.d.)

II Contextuality arguments in the sheaf approach
(Abramsky-Barbosa-Mansfield 2011, ABKLM 2015)

1. All-vs-nothing argument
2. Čech-cohomological argument
3. AvN-cohomology theorem
4. No-AvN example, a challenge to the approach
   (Abramsky-Barbosa-Carù-de Silva-Kishida-Mansfield 2017)

[Abramsky-Barbosa-Kishida-Lal-Mansfield 2015, 2016]

Non-Locality, Contextuality, and (Pre)sheaves



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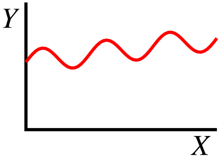

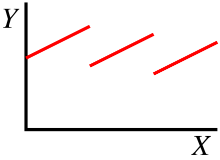

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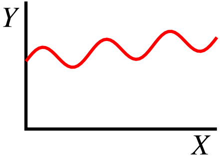

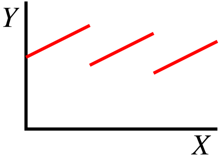

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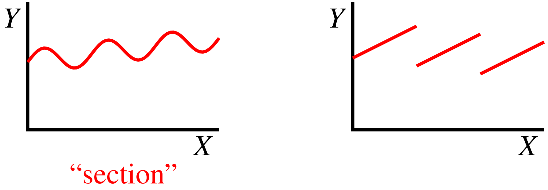

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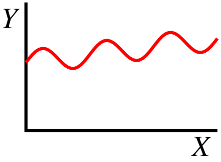

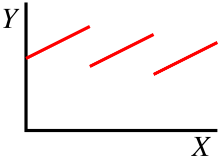

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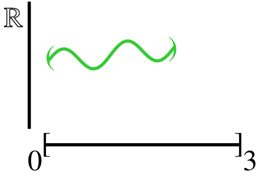

Continuous $f_1 : (0, 2) \to \mathbb{R}$

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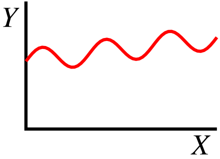

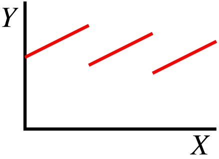

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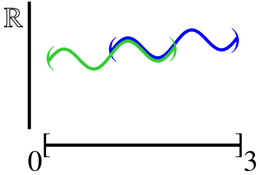

Continuous $f_1 : (0, 2) \to \mathbb{R}$ and
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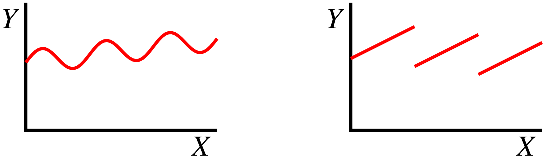

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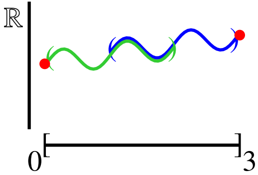

Continuous $f_1 : (0, 2) \to \mathbb{R}$ and $f_2 : (1, 3) \to \mathbb{R}$ extend to continuous $\bar{f} : [0, 3] \to \mathbb{R}$ uniquely.

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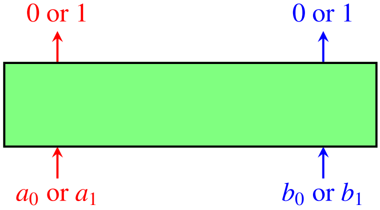

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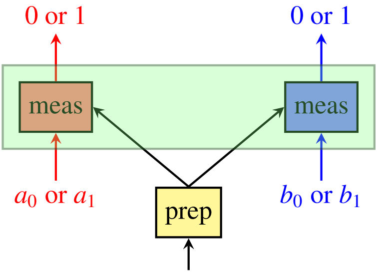

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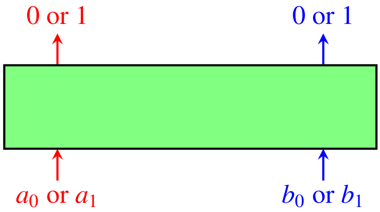

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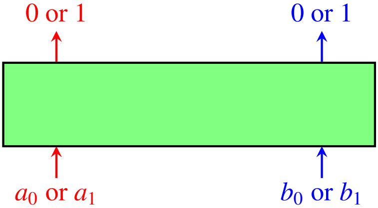

For each **context** (a_i, b_j), a distribution

$$p(o, o' \mid a_i, b_j)$$

over joint outcomes

$$(a_i, b_j) \mapsto (o, o')$$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$3/8$	$1/8$	$1/8$	$3/8$
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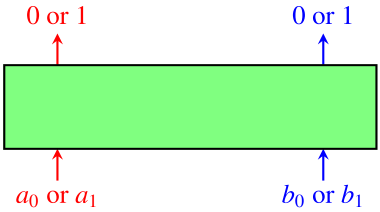

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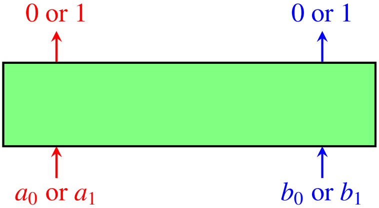

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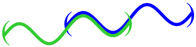

... having the same marginal

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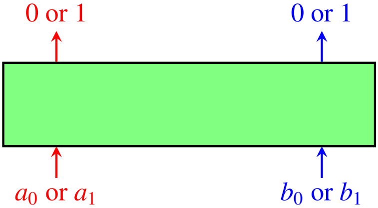

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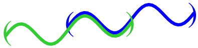

… having the same marginal = No-Signalling

Do these pieces of data over contexts extend to a distribution

$p(\cdot \mid a_0, a_1, b_0, b_1)$ for all measurements (a_0, a_1, b_0, b_1)

that gives back each $p(\cdot \mid a_i, b_j)$ as a marginal?

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(a_0, a_1, b_0, b_1)	p_1	p_2	$\cdots$	p_{15}	p_{16}

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Local causality $=_{\text{def}}$ Admits a hidden variable model

$\Longleftarrow$ / $\Longrightarrow$ Admits a deterministic hidden variable model

Factorizable

(Fine 1982, Abramsky-Brandenburger 2011)

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	(0, 0, 0, 0)	(0, 0, 0, 1)	⋯	(1, 1, 1, 0)	(1, 1, 1, 1)
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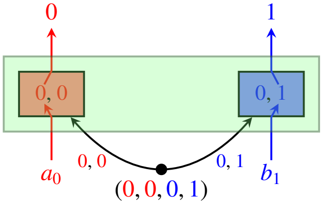

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"Possibility distributions"

Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
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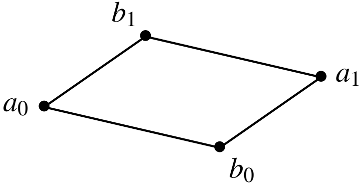

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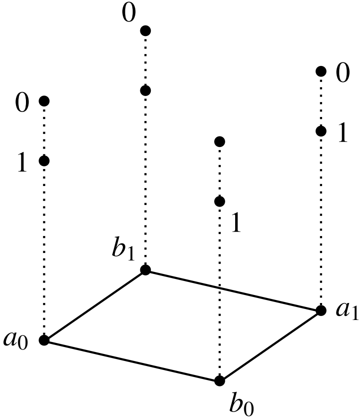

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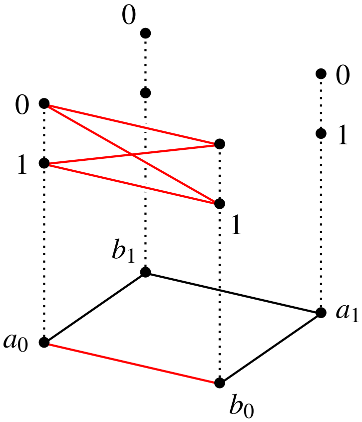

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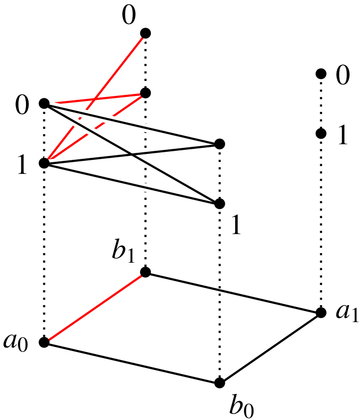

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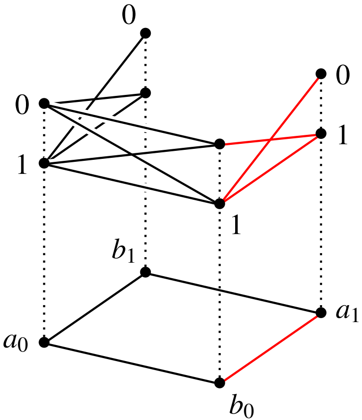

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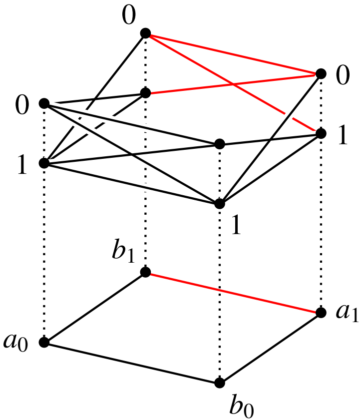

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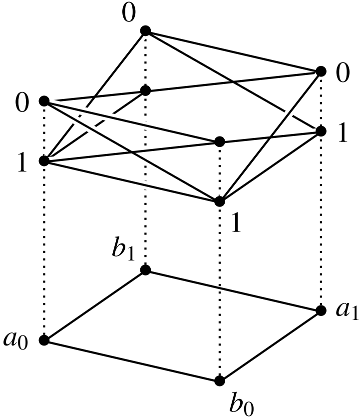

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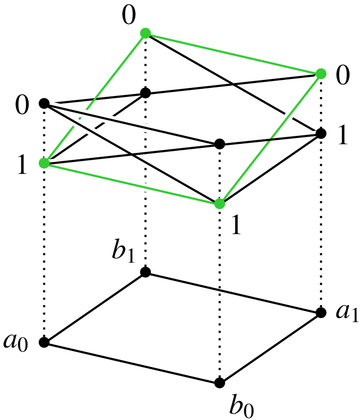

Some global sections, e.g.

$(a_0, a_1, b_0, b_1) \mapsto (1, 0, 1, 0)$;

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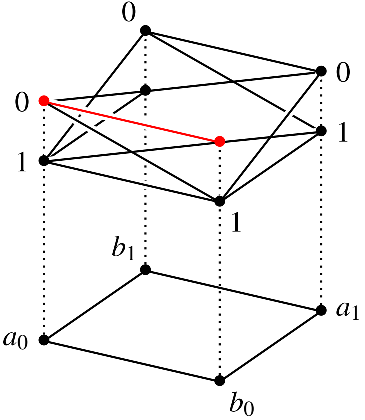
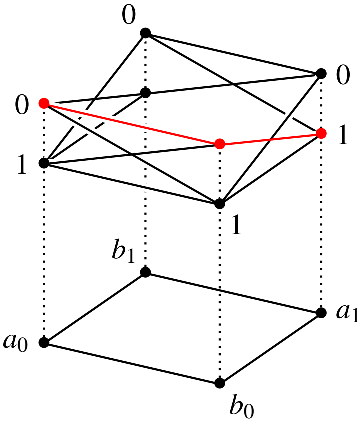
but . . .



"Possibility distributions"

Hardy 1993:

	00	01	10	11
$a_0 b_0$	✓	✓	✓	✓
$a_0 b_1$	0	✓	✓	✓
$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0



Some global sections, e.g.

$$(a_0, a_1, b_0, b_1) \mapsto (1, 0, 1, 0);$$

but . . .

"Possibility distributions"

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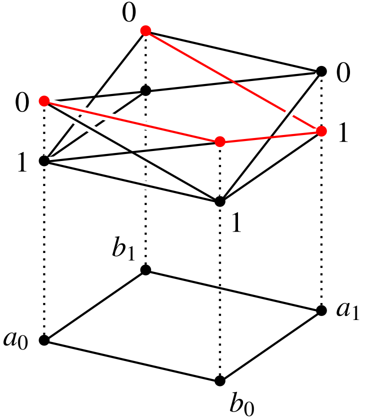
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0



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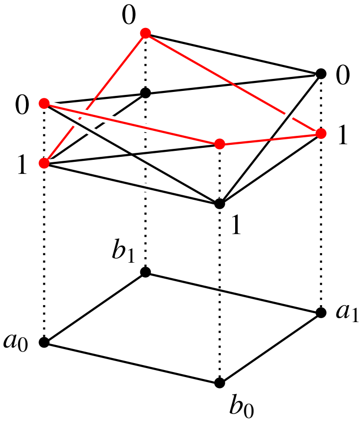
$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0



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$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0

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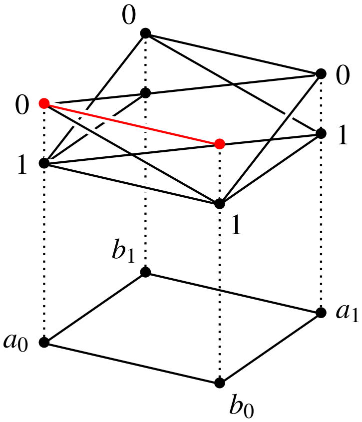
but . . .



"Possibility distributions"

Hardy 1993:

	00	01	10	11
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a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0

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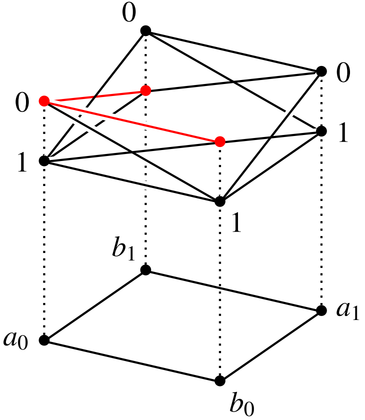

"Possibility distributions"

Hardy 1993:

	00	01	10	11
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$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0

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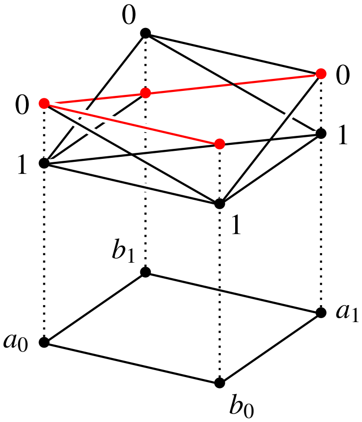
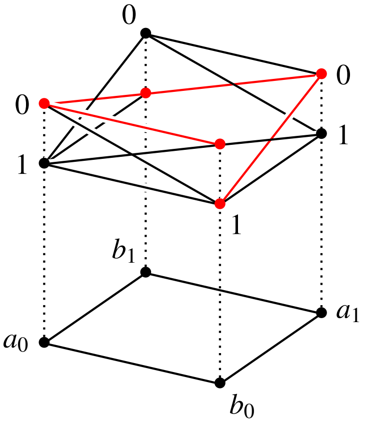


"Possibility distributions"

Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0



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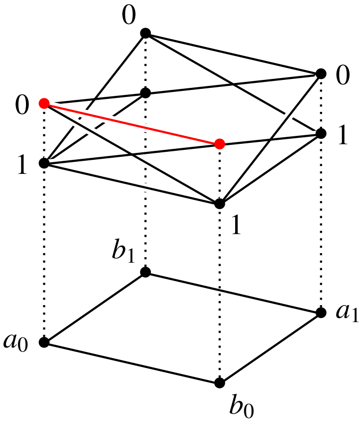
$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0



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but not all sections extend to global ones.

“Possibility distributions”

Hardy 1993:

	00	01	10	11
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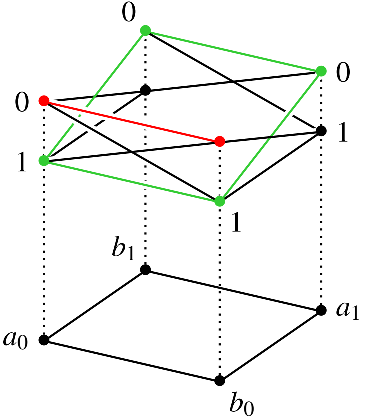
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0



Some global sections, e.g.

$(a_0, a_1, b_0, b_1) \mapsto (1, 0, 1, 0)$;

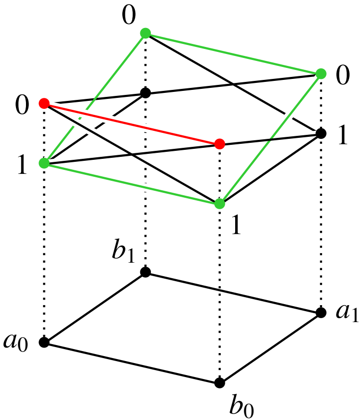
but not all sections extend to global ones.

	0000	⋯	0011	⋯	1001	1010	1011	⋯	1111
$a_0a_1b_0b_1$	0	⋯	✓	⋯	0	✓	✓	⋯	0

"Possibility distributions"

Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0



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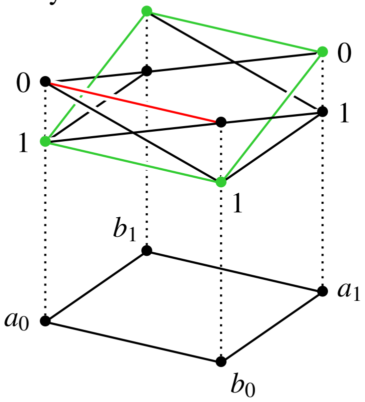
$(a_0, a_1, b_0, b_1) \mapsto (1, 0, 1, 0)$;

but not all sections extend to global ones.

	0000	$\cdots$	0011	$\cdots$	1001	1010	1011	$\cdots$	1111
$a_0a_1b_0b_1$	0	$\cdots$	✓	$\cdots$	0	✓	✓	$\cdots$	0

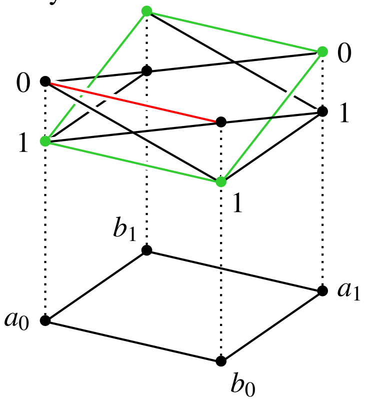
... **"Logical Non-Locality"**.

Hardy:



Logical non-locality: Not all sections extend to global ones.

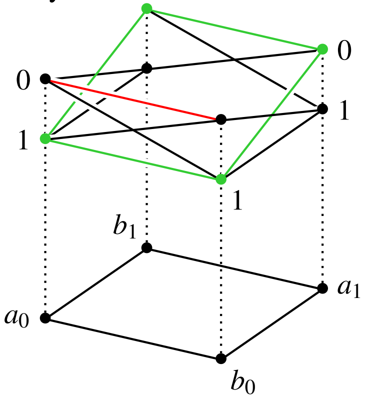
Hardy:



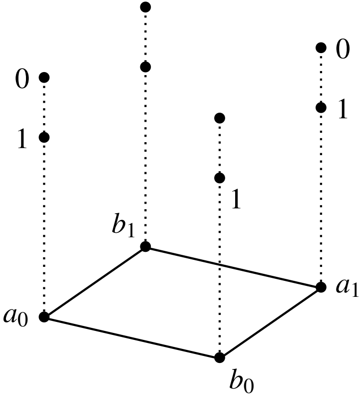
PR box:

Logical non-locality: Not all sections extend to global ones.

Hardy:

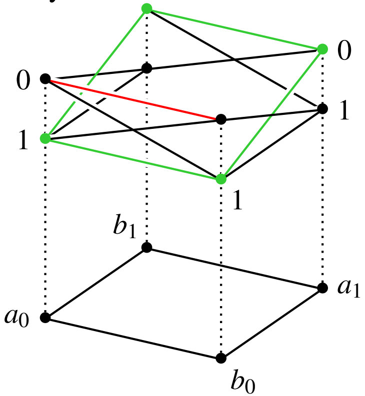


PR box:

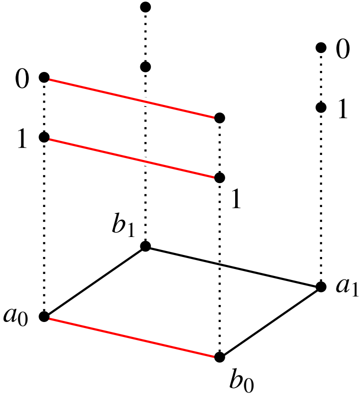


Logical non-locality: Not all sections extend to global ones.

Hardy:



PR box:



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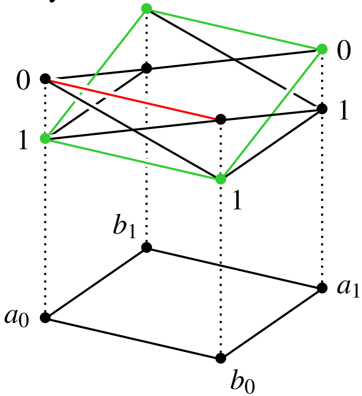
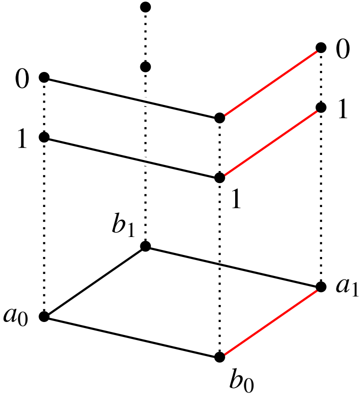


PR box:



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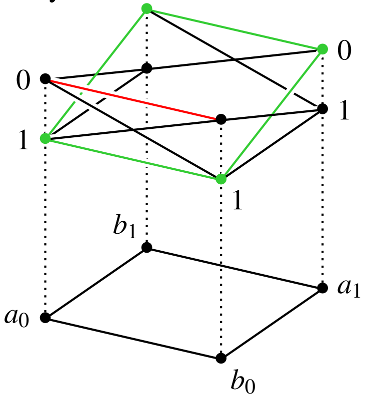

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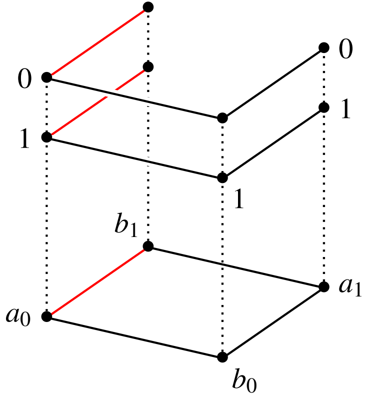

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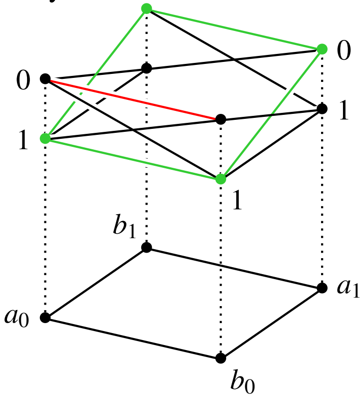

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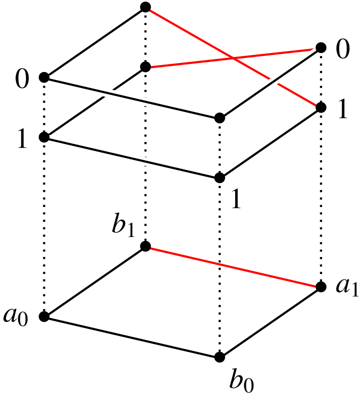

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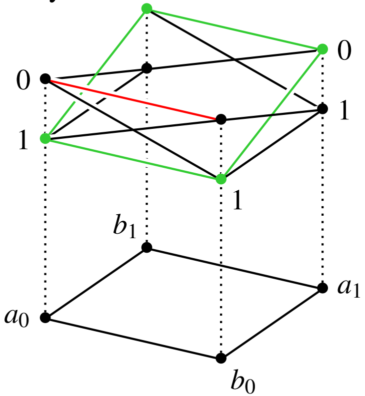
Hardy:



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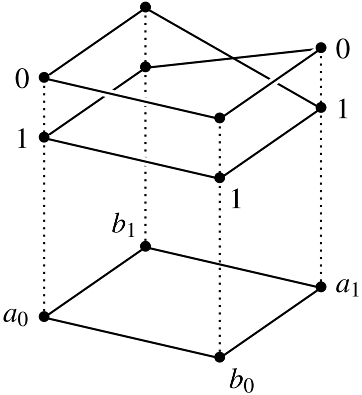

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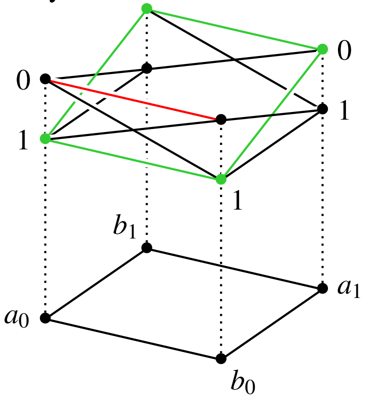

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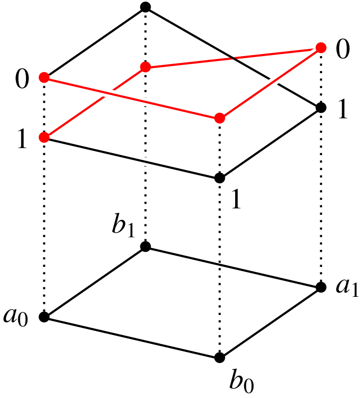

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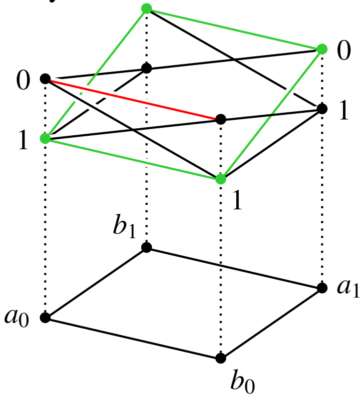

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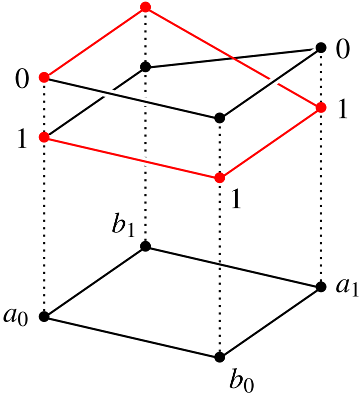

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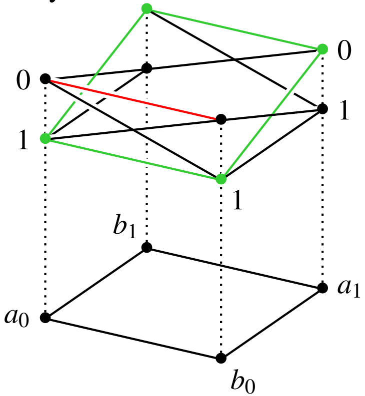

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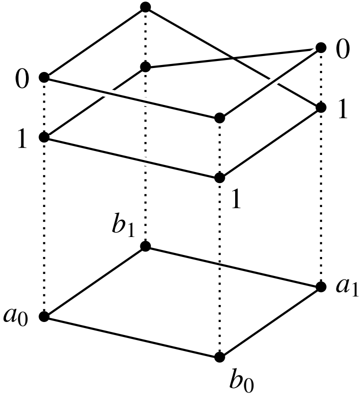
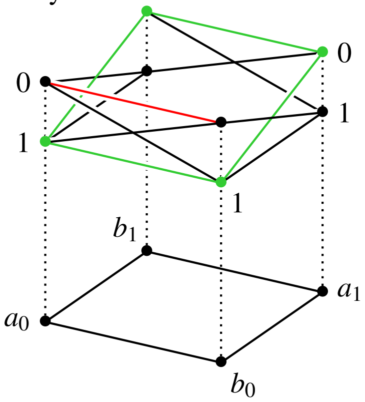


Logical non-locality: Not all sections extend to global ones.

Strong non-locality: No global section at all.

Hardy:



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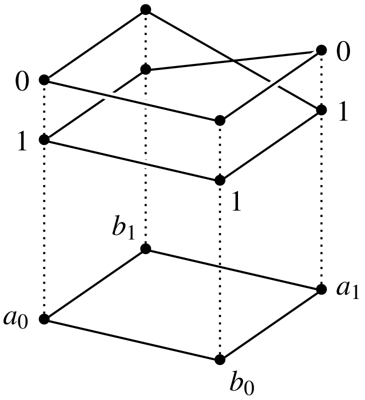
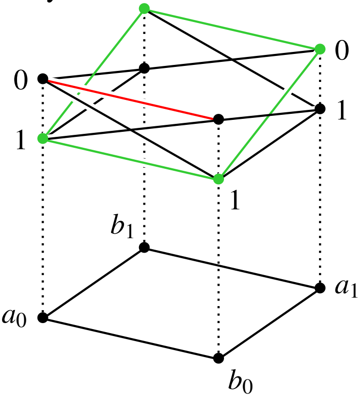


Logical contextuality: Not all sections extend to global ones.

Strong contextuality: No global section at all.

Hardy:



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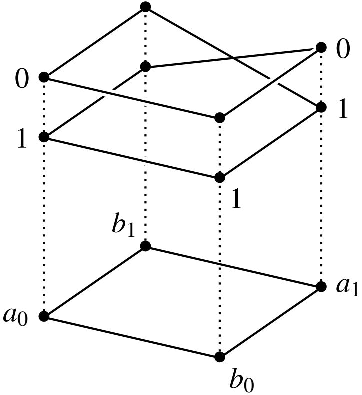


Logical contextuality: Not all sections extend to global ones.

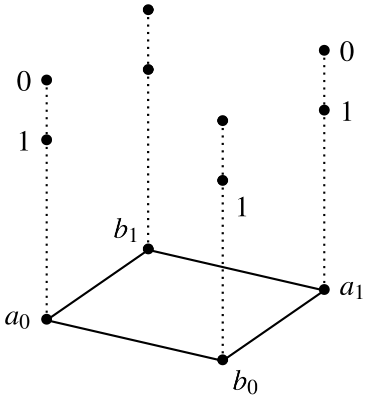
Strong contextuality: No global section at all.

Slogan:

Contextuality = Local consistency + global inconsistency

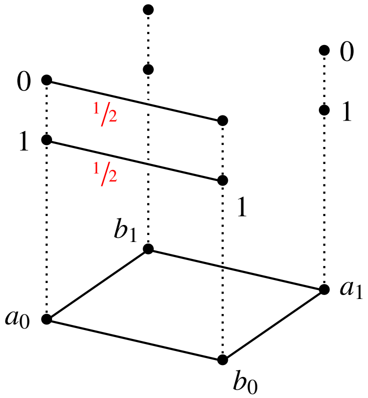
Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$



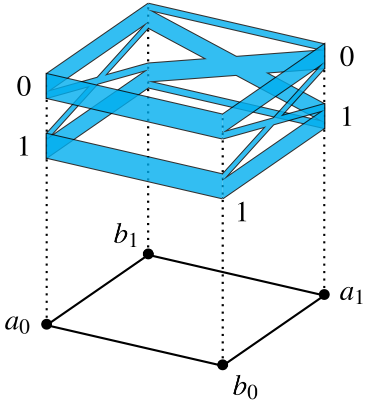
Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$



Bell vs. Hardy

	00	01	10	11
$a_0 b_0$	$1/2$	0	0	$1/2$
$a_0 b_1$	$3/8$	$1/8$	$1/8$	$3/8$
$a_1 b_0$	$3/8$	$1/8$	$1/8$	$3/8$
$a_1 b_1$	$1/8$	$3/8$	$3/8$	$1/8$



Bell vs. Hardy

	00	01	10	11
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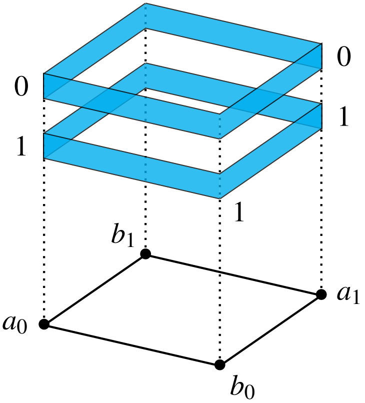

Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
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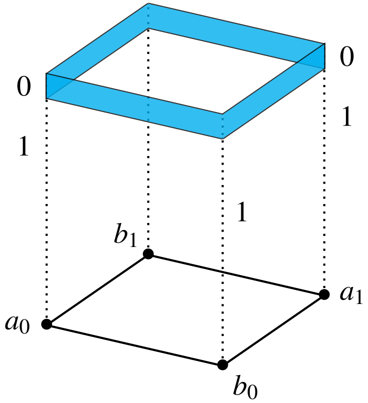
a_1b_1	$1/2$	0	0	$1/2$



Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
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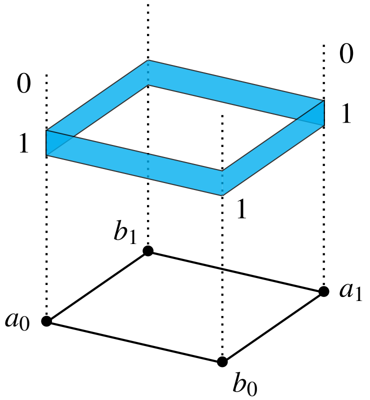

Bell vs. Hardy

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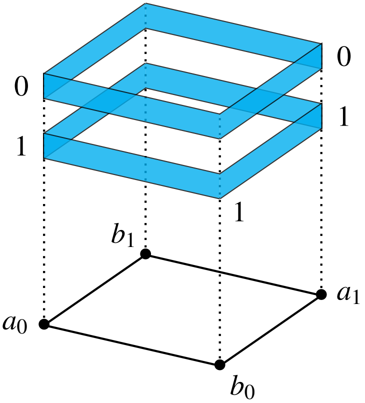
$a_0 b_0$	$1/2$	0	0	$1/2$
$a_0 b_1$	$1/2$	0	0	$1/2$
$a_1 b_0$	$1/2$	0	0	$1/2$
$a_1 b_1$	$1/2$	0	0	$1/2$



Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
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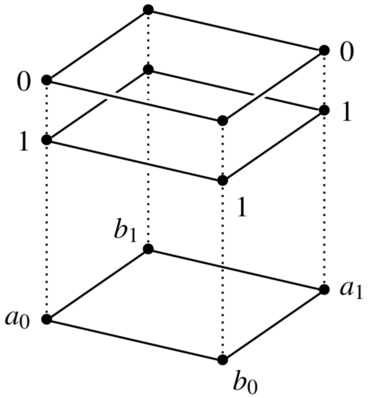

Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
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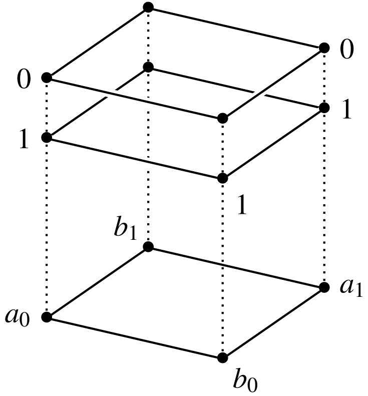
a_1b_1	$1/2$	0	0	$1/2$



$$\text{Bell local} \implies \text{Logically local,}$$
$$\text{Logically non-local} \implies \text{Bell non-local.}$$

Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
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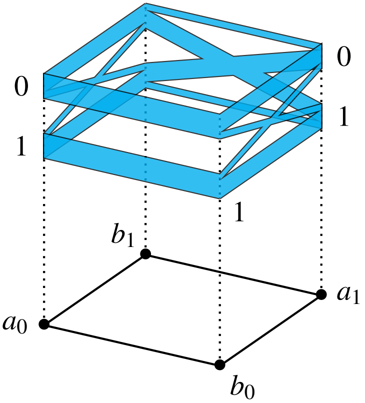
a_1b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$



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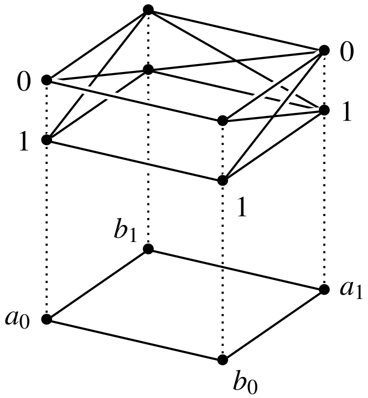
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$



$$\text{Bell local} \implies \text{Logically local,}$$
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Bell vs. Hardy

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a_0b_0	$1/2$	0	0	$1/2$
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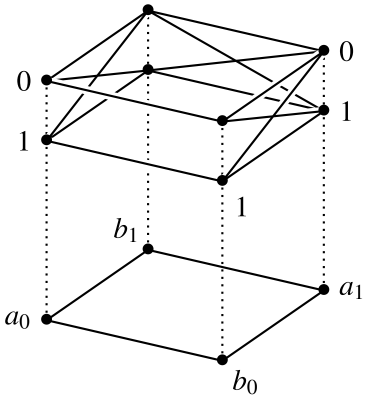

$$\begin{array}{rcl} \text{Bell local} & \Longrightarrow & \text{Logically local,} \\ \text{Logically non-local} & \Longrightarrow & \text{Bell non-local.} \\ & \not\Longleftarrow & \end{array}$$

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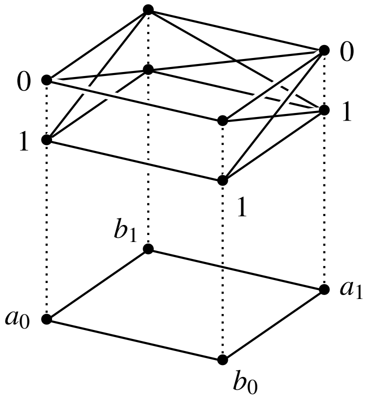

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Hieararchy of contextuality:

Probabilistic $\underset{\not\Rightarrow}{\Leftarrow}$ **Logical** $\underset{\not\Rightarrow}{\Leftarrow}$ **Strong contextuality**

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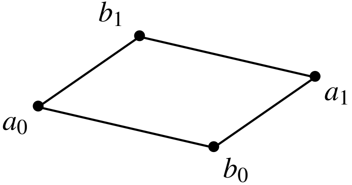

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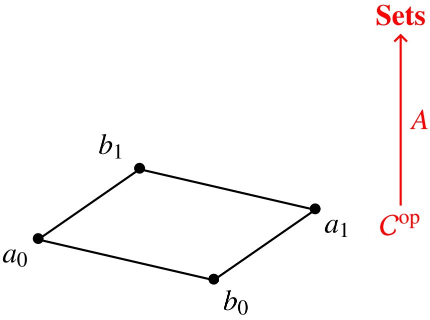

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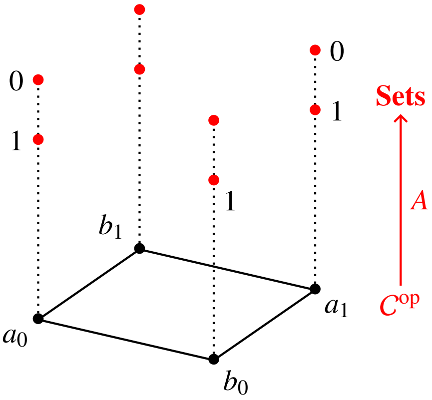

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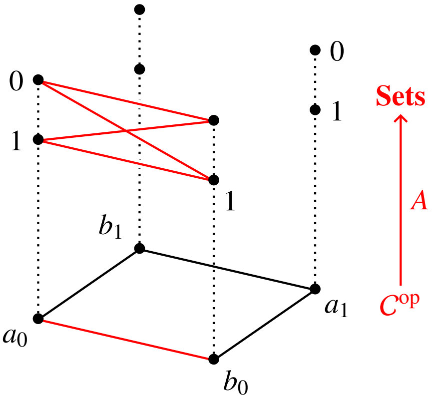

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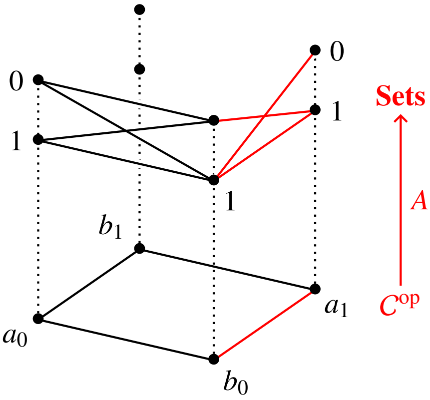

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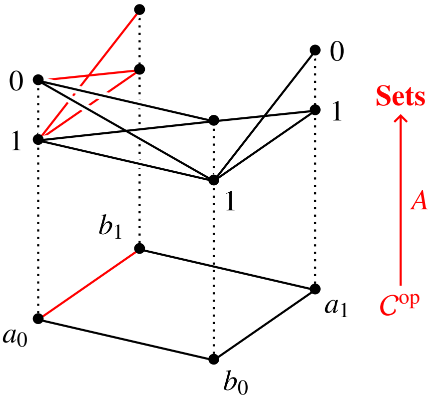

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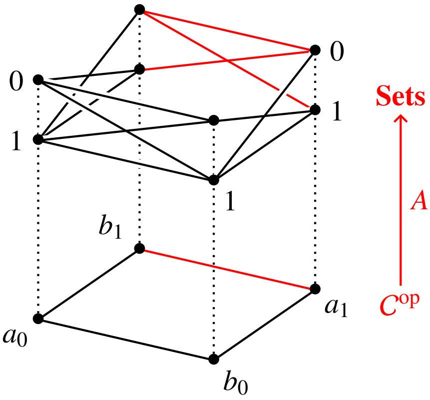

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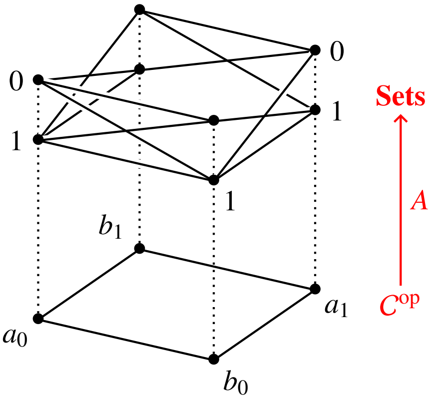

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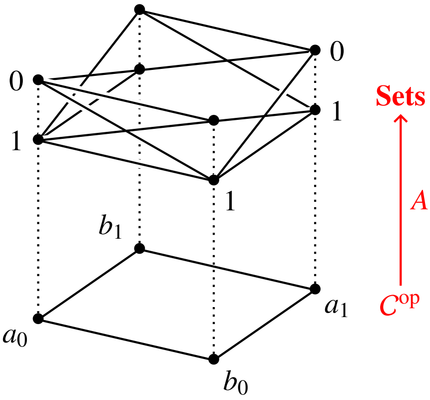

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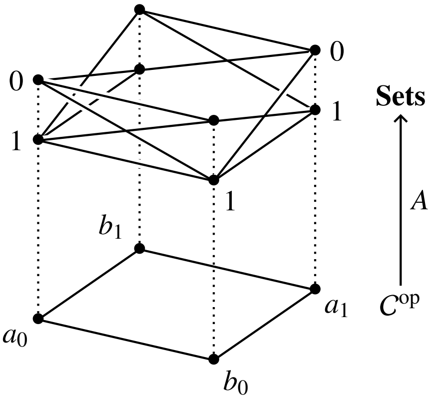

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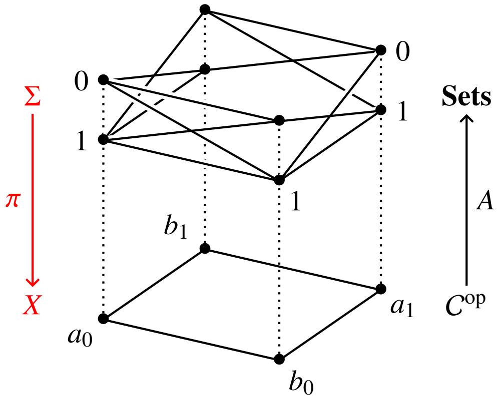

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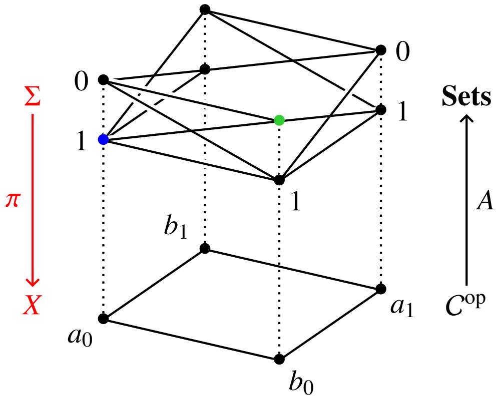

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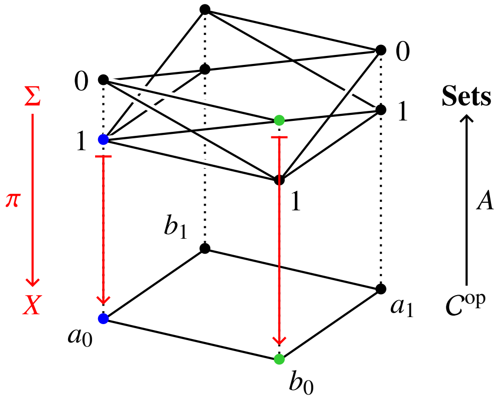

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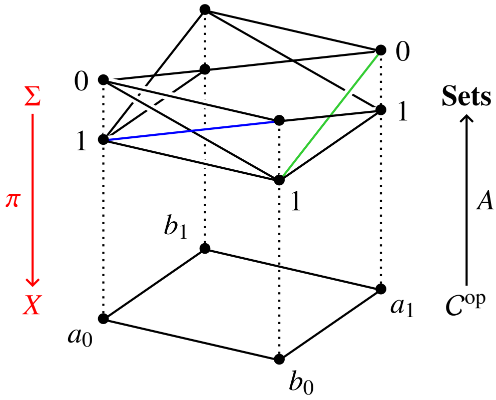

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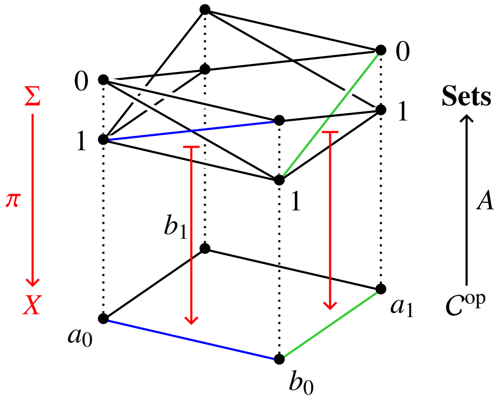

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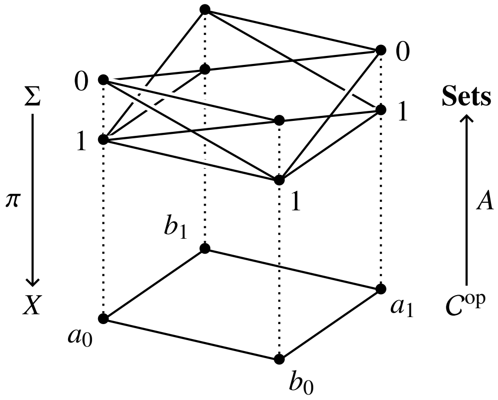

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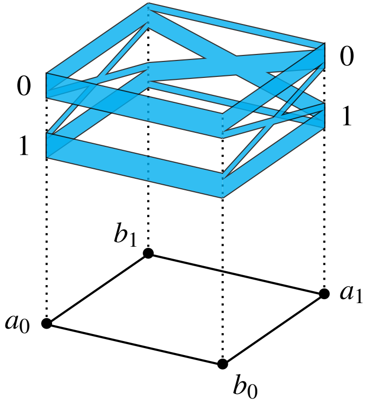

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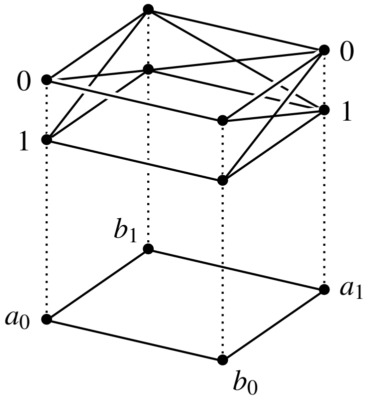

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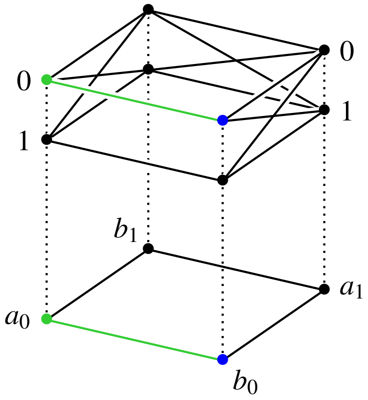

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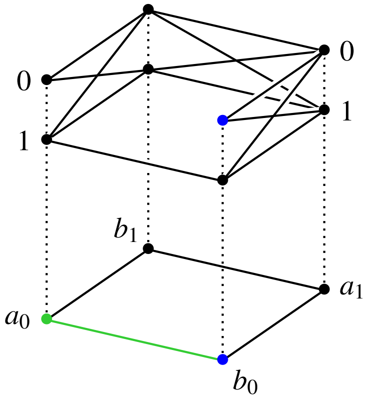

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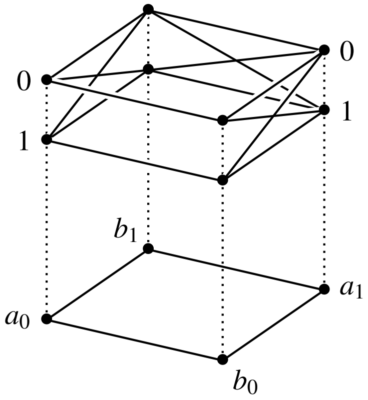

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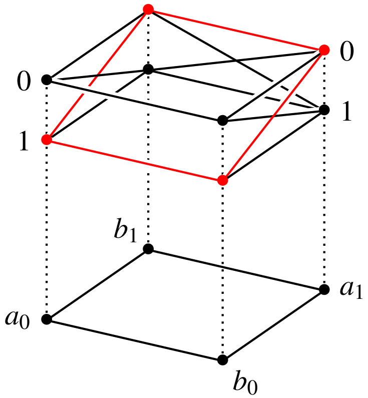

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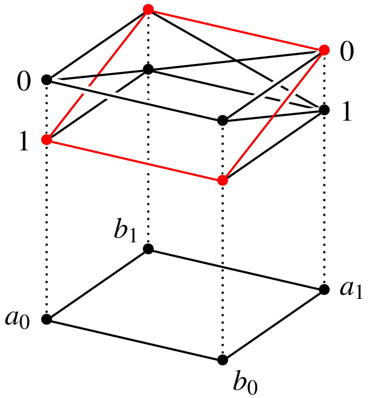

Relation to Other Expressions

Local and no-signalling polytopes

Convex combinations of tables are again tables;
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Local tables (i.e. ones that admit hidden variable models) form a **polytope** whose vertices are exactly the deterministic tables:

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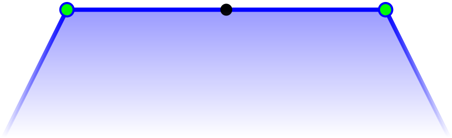

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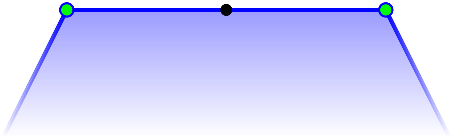
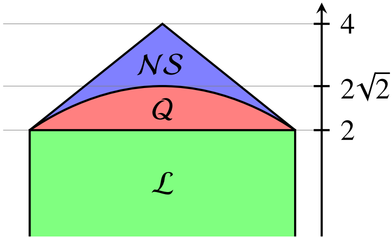


No-signalling tables then form a larger polytope.

The polytopes are bounded by the CHSH and other inequalities.
E.g. in $(2, 2, 2)$,



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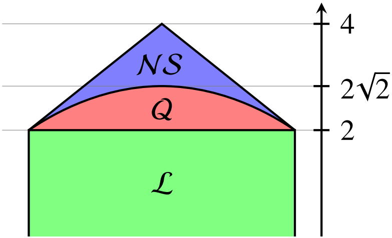

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—They capture the “combinatorial” structure of the polytope $\mathcal{NS}$:

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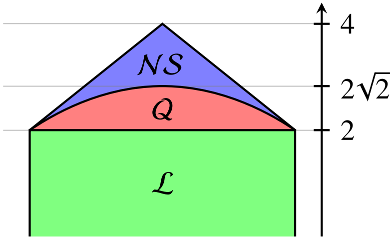

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Theorem (Abramsky-Barbosa-KK-Lal-Mansfield 2016)**.**

Take the supports of probabilistic tables in $\mathcal{NS}$, and order them by context-wise inclusion of supports. Then the poset obtained is isomorphic to the face lattice of $\mathcal{NS}$.

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To this theory, an "ontological model" adds:

- a set Ω of "ontic states" λ,
- for each p, a distribution $\mu_p \in \mathcal{D}(\Omega)$,
- for each t, a map $\Gamma_t : \Omega \to \mathcal{D}(\Omega)$,
- for each m, a map $\xi_m : \Omega \to \mathcal{D}(O)$.

Spekkens' notion of measurement contextuality

(due to Linde Wester and Shane Mansfield)

An "operational theory" talks about:

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The model reproduces the theory if

$$\Pr(k \mid p, t, m) = \int d\lambda' d\lambda \; \xi_m(\lambda')(k) \; \Gamma_t(\lambda)(\lambda') \; \mu_p(\lambda).$$

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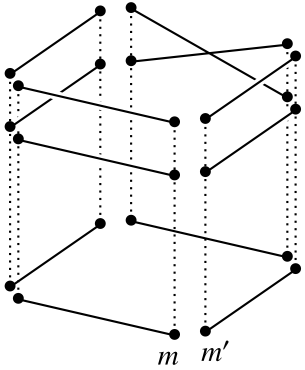

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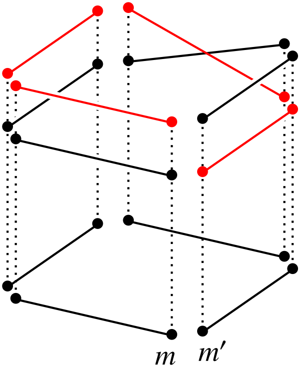

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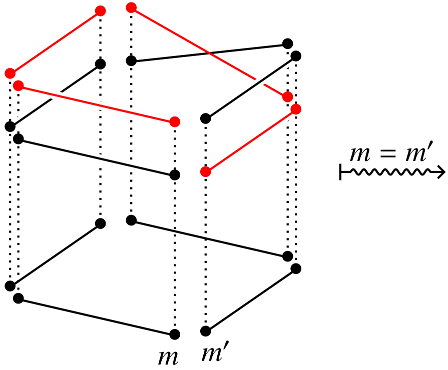

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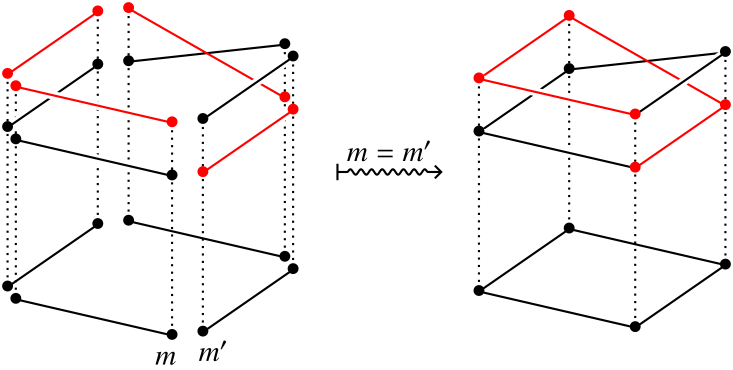

All-vs-Nothing Argument

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Joint outcomes may satisfy parity equations, e.g.:

$(0, 0) \vDash x \oplus y = 0$
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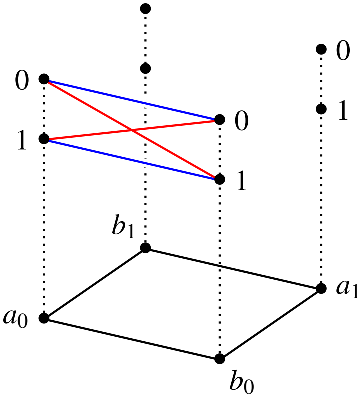

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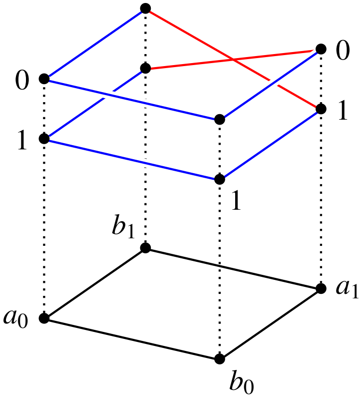

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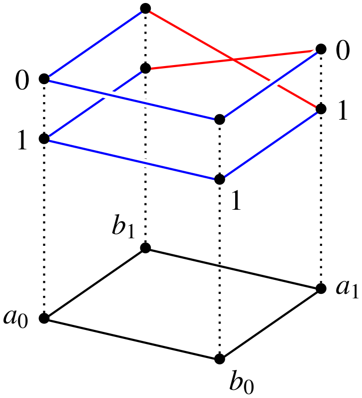

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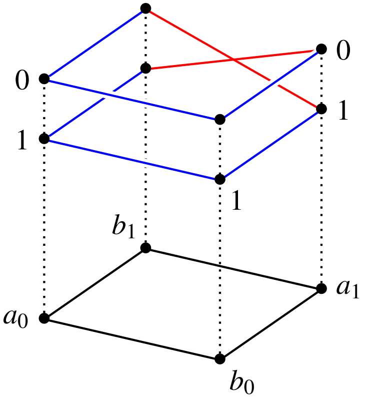

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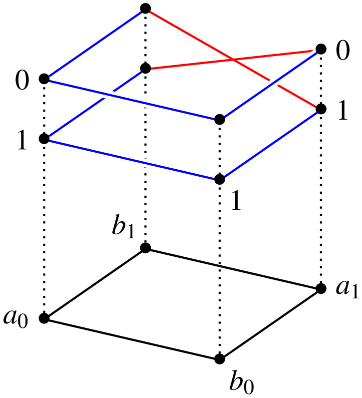

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i.e. no global assignment consistent with the constraints,
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An empirical model is strongly contextual if it "admits" generalized AvN argument, meaning that its sections satisfy linear equations that are inconsistent in the way above.

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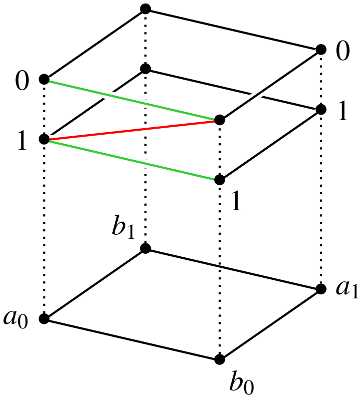

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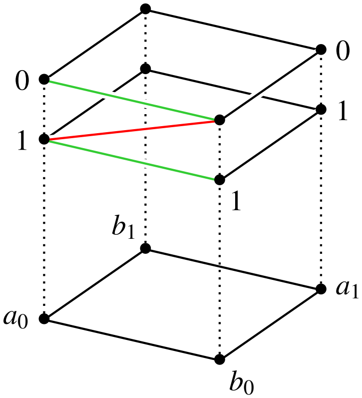

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No global assignment
(consistent with the other
constrants) satisfies $\color{red}{a_0 \oplus b_0 = 1}$,
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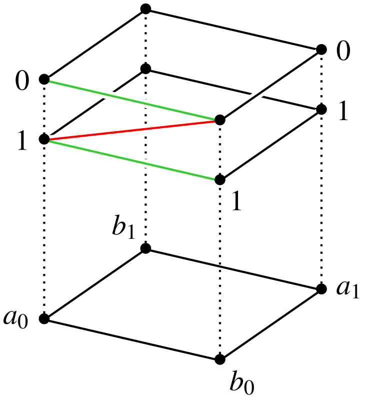

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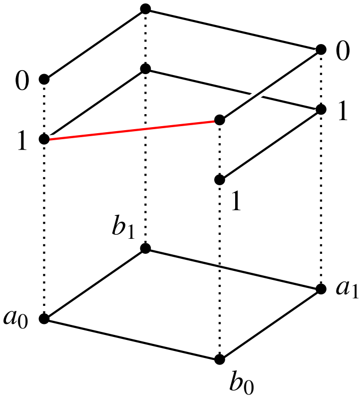

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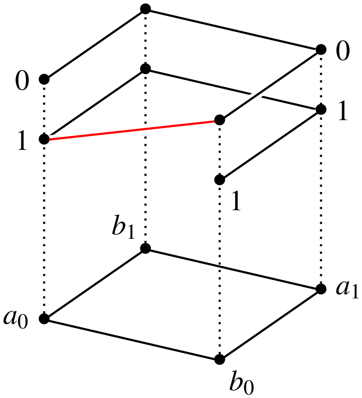

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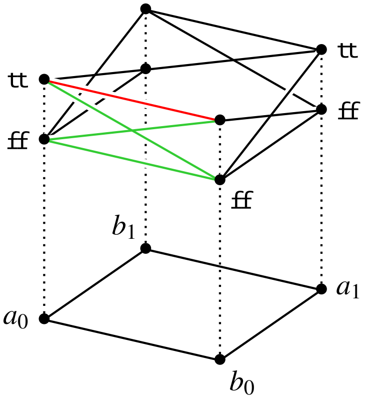

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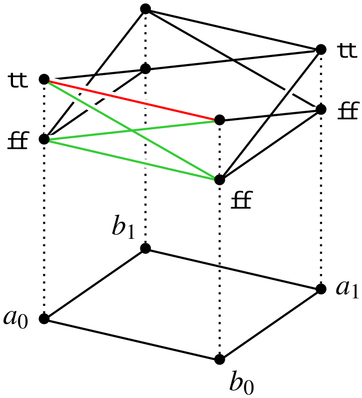

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Not just linear equations, we may use other vocabulary;
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—But linear equations are nice.

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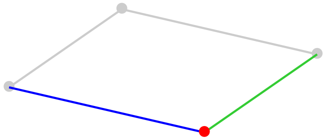

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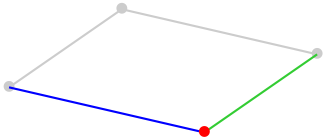

So we are now taking a new simplicial complex, with

$$\begin{array}{ll} U \in C & \text{as vertices,} \\ (U, V) \in NC^1 & \text{as edges.} \end{array}$$

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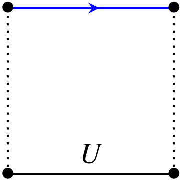

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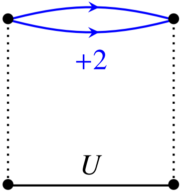

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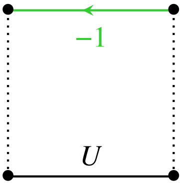

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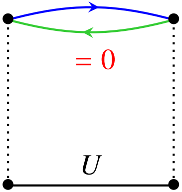

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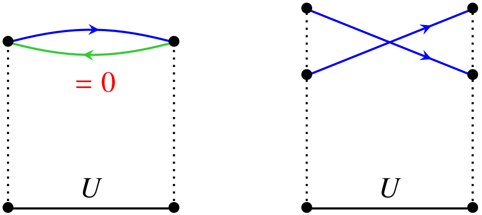

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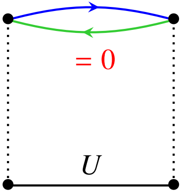

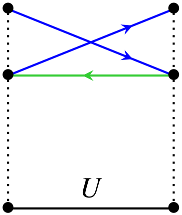

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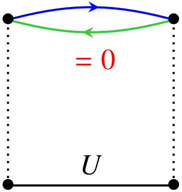

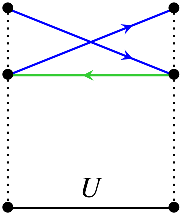

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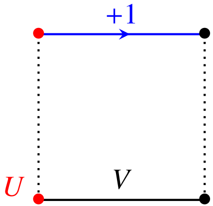

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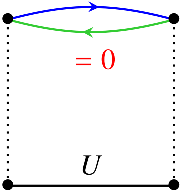

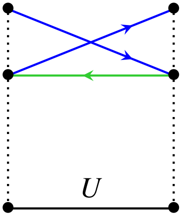

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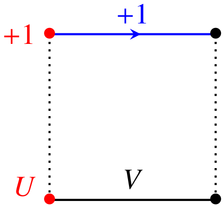

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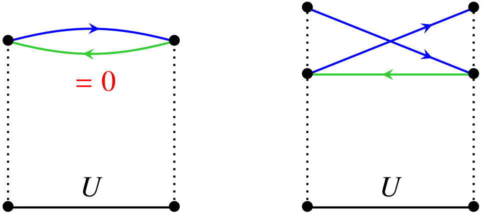

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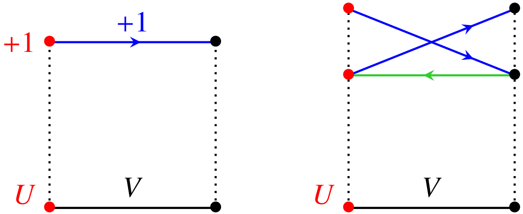

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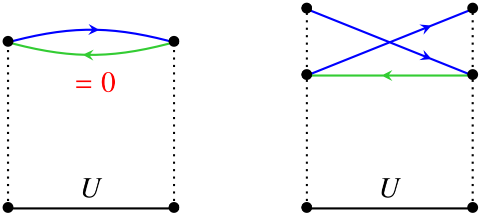

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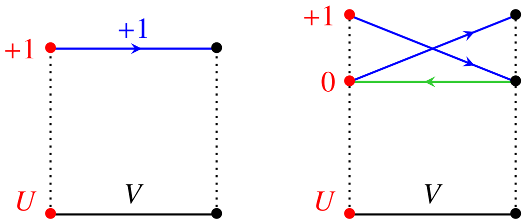

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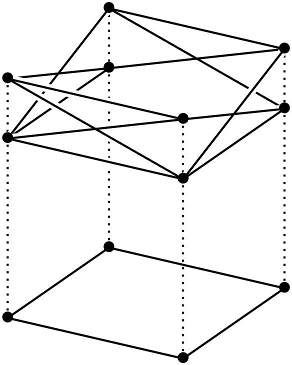

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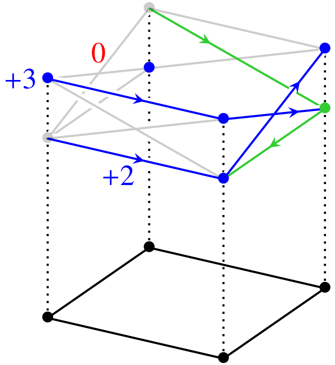

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$$\omega \in \prod_{U \in \mathcal{C}, U \neq \emptyset} F(U)$$

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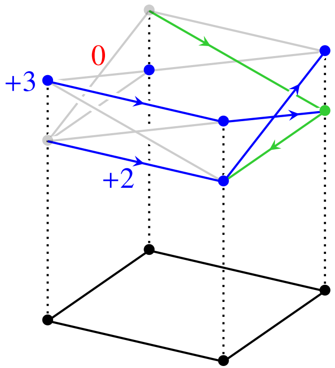

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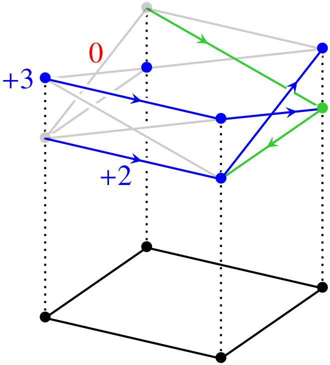

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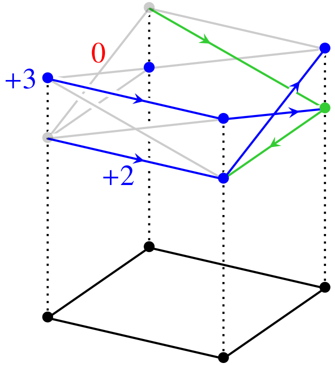

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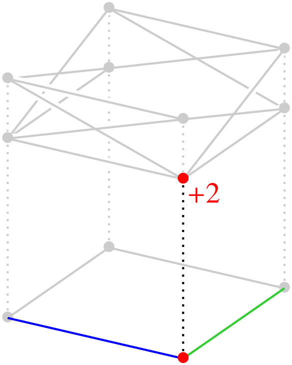

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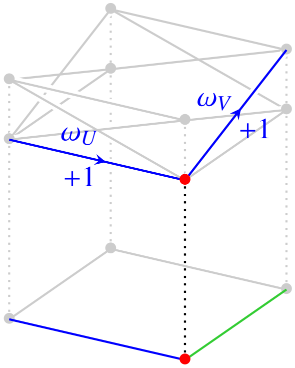

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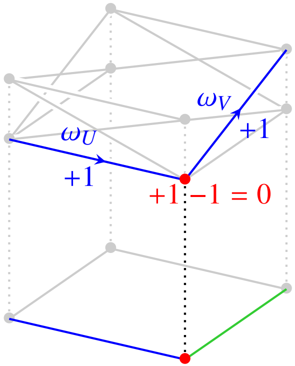

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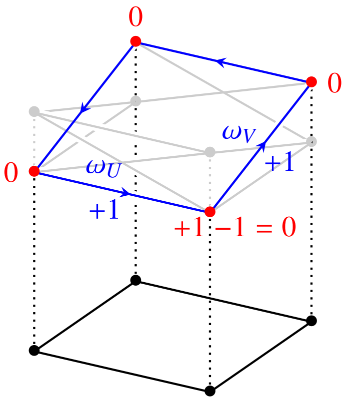

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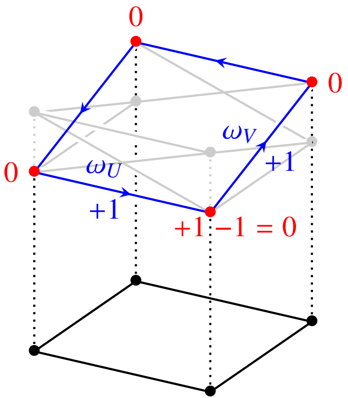

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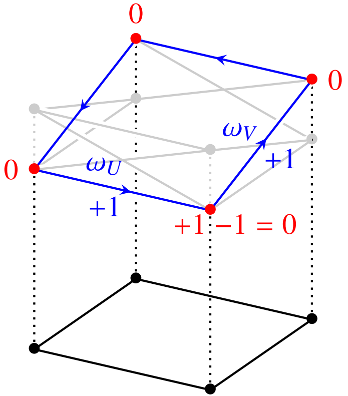

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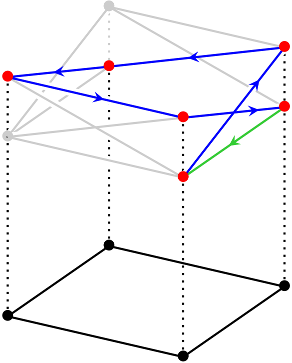

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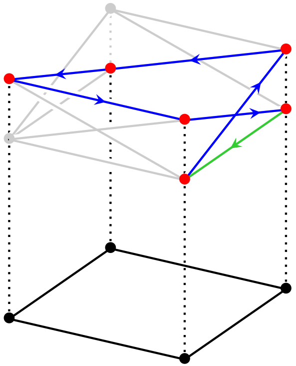

Caveat:

global section

$\underline{\cap}$ $\not\underline{\cup}$

0-cocycle

The group of 0-cocycles, i.e. $\ker(\delta^0)$, is written $\check{H}^0(\mathcal{C}, F)$.

❼

1-cochains

$$C^1(C, F)$$

$$\ker \delta^0 \rightarrowtail C^0(C, F) \xrightarrow{\delta^0} C^1(C, F)$$

0-cocycles $\ker \delta^0 = \check{H}^0(C, F)$

0-cochains $C^0(C, F)$

❼

$$\begin{array}{cccccc}
 & & & \text{1-cochains} & & \text{2-cochains} \\
 & & & C^1(C,F) & \xrightarrow{\ \delta^1\ } & C^2(C,F) \\
 & & \overset{\delta^0}{\nearrow} & & & \\
\text{0-cocycles} & & & & & \\
\ker\delta^0 & \rightarrowtail & C^0(C,F) & & & \\
= \check{H}^0(C,F) & & \text{0-cochains} & & &
\end{array}$$

❼

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 & & & \nearrow^{\delta^0} & C^1(C,F) & \xrightarrow{\delta^1} & C^2(C,F) \\
\text{0-cocycles} & & & & \cup & & \\
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\color{red}{\ker \bar{\delta}^0} & \color{red}{\rightarrowtail} & C^0(C,F) & \color{red}{\xrightarrow[\bar{\delta}^0]{}} & Z^1(C,F) & \color{red}{\twoheadrightarrow} & \color{red}{\operatorname{coker} \bar{\delta}^0} \\
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$$

❼

1-cochains, 2-cochains, 0-cocycles

$$\begin{array}{ccccccc} & & & & C^1(C,F) & \xrightarrow{\delta^1} & C^2(C,F) \\ & & & \nearrow^{\delta^0} & \uparrow & & \\ \ker\bar{\delta}^0 & \rightarrowtail & C^0(C,F) & \xrightarrow[\bar{\delta}^0]{} & Z^1(C,F) & \twoheadrightarrow & \operatorname{coker}\bar{\delta}^0 \\ = \check{H}^0(C,F) & & & & & & = \check{H}^1(C,F) \end{array}$$

0-cochains, 1-cocycles

❽ For an answer to "Does $s : U \to R$ extend to a global section?",

❼

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\begin{array}{ccccccc}
 & & & & \text{1-cochains} & & \text{2-cochains} \\
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 & & & \overset{\delta^0}{\nearrow} & \uparrow & & \\
\text{0-cocycles} & & & & & & \\
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\text{0-cocycles} & & & \overset{\delta^0}{\nearrow} & \cup\!\uparrow & & \\
\color{red}{\ker \bar{\delta}^0} & \color{red}{\rightarrowtail} & C^0(C,F) & \color{red}{\xrightarrow[\bar{\delta}^0]{}} & Z^1(C,F) & \color{red}{\twoheadrightarrow} & \color{red}{\operatorname{coker} \bar{\delta}^0} \\
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7

$$
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$$\mathbf{0} \longrightarrow F_{\bar{U}} \rightarrowtail F \xrightarrow{\;p\;} \!\!\!\!\twoheadrightarrow F{\restriction}_U$$

❾

$$
\begin{array}{ccccccccc}
 & & \check{H}^0(C, F_{\bar{U}}) & \longrightarrow & \check{H}^0(C, F) & \longrightarrow & \check{H}^0(C, F|_U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & C^0(C, F_{\bar{U}}) & \longrightarrow & C^0(C, F) & \longrightarrow & C^0(C, F|_U) & \longrightarrow & \mathbf{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & Z^1(C, F_{\bar{U}}) & \longrightarrow & Z^1(C, F) & \longrightarrow & Z^1(C, F|_U) & \longrightarrow & \mathbf{0} \\
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$$\check{H}^0(C,F|_U) \longrightarrow \check{H}^1(C,F_{\bar{U}})$$

⑨

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 & & \check{H}^0(C,F_{\bar{U}}) & \longrightarrow & \check{H}^0(C,F) & \longrightarrow & \check{H}^0(C,F|_U) & & \\
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So we have an exact sequence:

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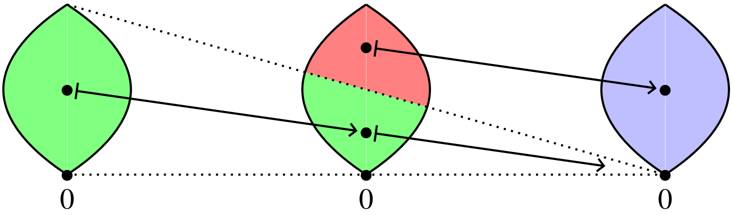

9

$$
\begin{array}{ccccccccc}
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\end{array}
$$

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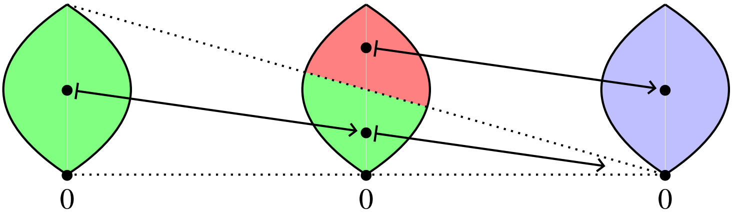
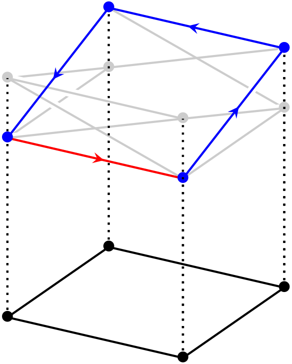
$$\text{0-cocycles} \xrightarrow{\text{restriction}} F(U) \xrightarrow{\gamma} \check{H}^1(C, F_{\bar{U}})$$



Čech-cohomological test for contextuality:

Each section $s \in A_U \subseteq F(U)$ has the "obstruction" $\gamma(s)$:

$$s \text{ extends to a cocycle} \iff \gamma(s) = 0.$$



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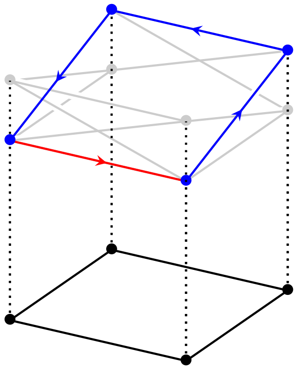

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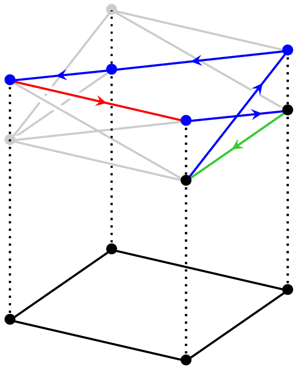
- False positives,
  e.g. in Hardy model:



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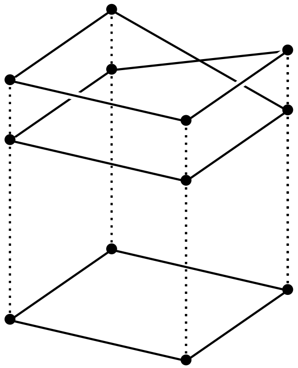
- False positives,
  e.g. in Hardy model.
- Works for many cases;
  e.g. PR box:



AvN-Cohomology Theorem

In fact, this cohomological test works for all the previously known examples of strong contextuality (GHZ, Kochen-Specker, . . .).

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"Strongly contextual by AvN argument"
$\Longrightarrow$ "Strongly contextual by cohomology":

Theorem (Abramsky, Barbosa, KK, Lal, Mansfield 2015)**.**
Let $\mathcal{M}$ be a model over $\mathcal{C}$. Then

- $\mathcal{M}$ admits a generalized AvN argument in a ring R

implies

- In Čech cohomology (using R), no section s in $\mathcal{M}$ has $\gamma(s) = 0$.

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Hence a hieararchy of strong contextuality:

$$\text{AvN} \underset{\not\Leftarrow}{\Rightarrow} \text{gen. AvN} \underset{\not\Leftarrow}{\Rightarrow} \text{cohom. SC} \underset{\not\Leftarrow}{\Rightarrow} \text{SC}$$

To prove this theorem, first observe:

Elements of the free module $F(R^U)$ are
"formal" linear combinations of sections $s_i : U \to R$,

$$\alpha_1 s_1 + \cdots + \alpha_n s_n \quad \text{for} \quad \alpha_i \in R.$$

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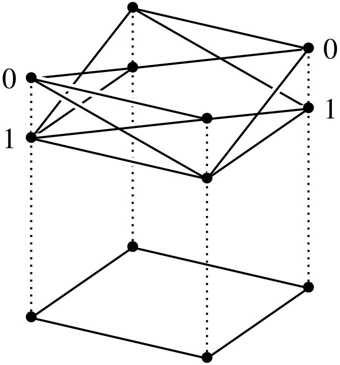
Def. Given an empirical model A, define a new one $\mathrm{aff}(A)$ by

$$\mathrm{aff}(A)_U = \mathrm{aff}(A_U).$$

E.g.



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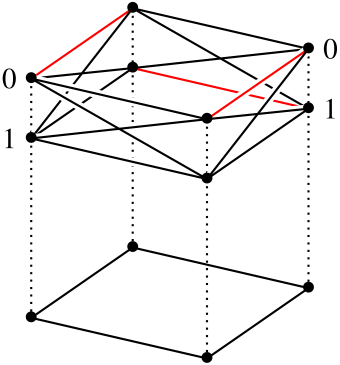

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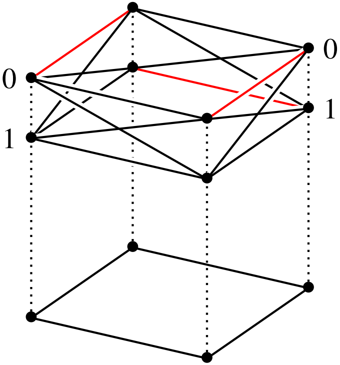

$$\color{red}{(0,0)} = (0,1) \oplus (1,0) \oplus (1,1)$$

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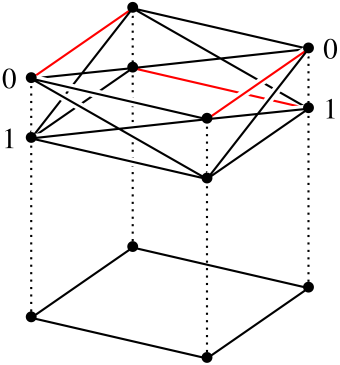

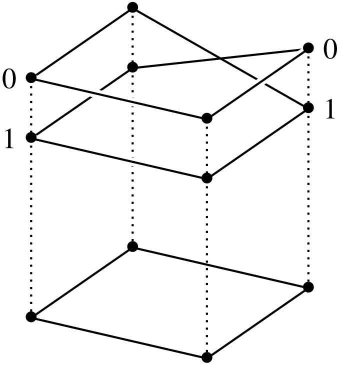

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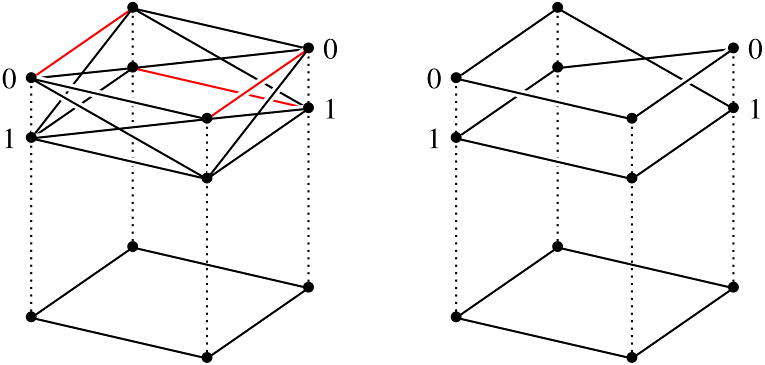

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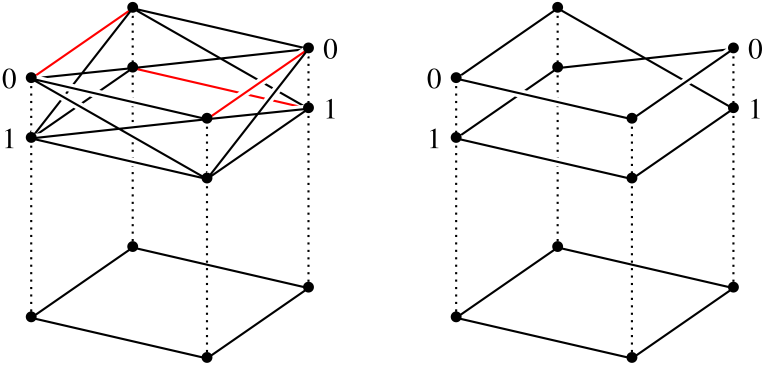

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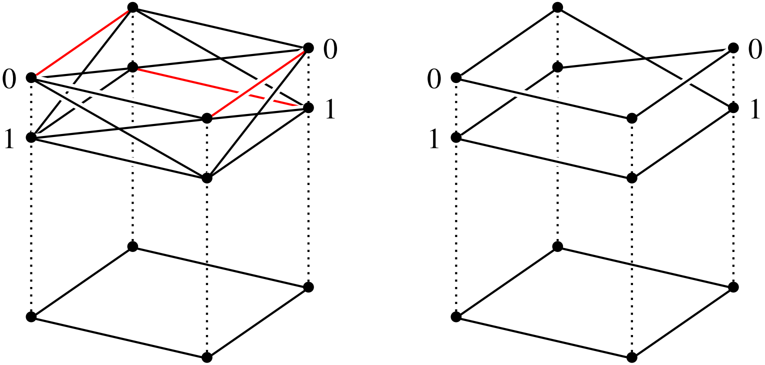

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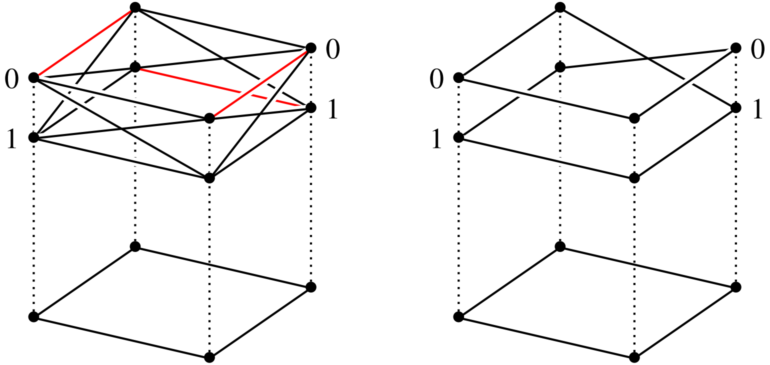

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Lemma. If a section $s \in R^U$ is the U-component of a matching family $\omega = \{\, \omega_V \in F(A_V) \,\}_{V \in C}$ (i.e. $1 \cdot s = \omega_U$), then $\epsilon \circ \omega = \{\, \epsilon_V(\omega_V) \in R^V \,\}_{V \in C}$ is a global section of aff(A) with the U-component s.

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Moreover, for this state to exhibit strong non-locality,
only the “equatorial” measurements, i.e. local measurements
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Local measurements $x_i = a_0, \ldots, a_{2n-1}, b_0, \ldots, b_{2n-1}, c_0, c_n$:

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Proof. Any global assignment of values to all x_i must satisfy, for all $i, j < 2n$, both

$$\begin{aligned} &\tfrac{i\pi}{2n} + a_i\pi + \tfrac{j\pi}{2n} + b_j\pi + c_0\pi \neq \pi \mod 2\pi, \\ &\tfrac{i\pi}{2n} + a_i\pi + \tfrac{j\pi}{2n} + b_j\pi + (-1)^{c_n}\tfrac{\pi}{2n} \neq \pi \mod 2\pi. \end{aligned}$$

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